Motion of a bore over a sloping beach

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The results of numerical calculations are presented for the motion of a bore over a uniformly sloping beach. The shallow water equations are solved in finite difference form, and a technique is developed for fitting in the bore at each step. The results are compared with the approximate formula given by Whitham (1958) and close agreement is found. The approximate theory is considered further here; the main addition is a rigorous proof that, within the shallow water theory, the height of the bore always tends to zero at the shoreline.

1. Introduction

This paper gives an account of numerical solutions of the shallow water equations for the motion of a bore on a uniformly sloping beach. In a previous paper (Whitham 1958), an approximate formula was derived for the variation in the strength and height of the bore. The predictions were seen to contain specially interesting features, one of the main ones being that the height of the bore always tends to zero as it approaches the shoreline. Indeed, in the earlier account these predictions were viewed with a certain amount of suspicion, even though checks of the analogous theory in certain problems of gas dynamics show extremely high accuracy. Subsequently, it was realized that the predicted behaviour is correct† (within the shallow water theory at least) and the result that the bore height tends to zero at the shoreline can be proved rigorously. This proof, together with recapitulation and further discussion of the approximate method, is given in § 2.

The numerical investigation confirms the previous results completely. In fact the agreement is quite remarkable; the differences are of the same order as the probable errors in the numerical work. In addition to this check, the numerical solution provides the full details of the flow behind the bore; these were not obtained previously. Furthermore, the solution is found for various starting conditions whereas it is assumed in the approximate theory that the bore is initially moving with constant speed and height in water of uniform depth. For these other starting conditions, it is found that the solutions eventually settle down and follow the approximate formula; the appropriate choice of the arbitrary constant factor in that formula varies considerably with the starting conditions for the same initial bore strength, but the functional dependence near the shoreline is the same. A similar insensitivity to the detailed initial conditions is well known for the analogous problem of the converging cylindrical shock in gas dynamics. The numerical results are discussed in detail in § 4.

† In this connexion we are grateful for discussions with G. F. Carrier and H. P. Greenspan.

For this problem, it was decided to develop a numerical method which uses fixed equal space intervals rather than the characteristics method which is not well suited to machine calculations. Stable ways of differencing the equations are well known, and the only question is how to fit in the bore at each step. This requires special care, but the method adopted here seems to be satisfactory in every respect. The details of the numerical scheme are given in § 3. It is planned to extend this method for use in the analogous shock-wave problems in gas dynamics.

2. Approximate formula and behaviour near the shoreline

If $h(x,t) = h_0(x) + \eta(x,t)$ denotes the depth of the water, where $h_0(x)$ is the undisturbed depth ahead of the bore, and $u(x,t)$ is the particle velocity, the equations of the shallow water theory are

$$\eta_t + \{(h_0+\eta)\,u\}_x = 0, \tag{1}$$

$$u_t + uu_x + g\eta_x = 0. \tag{2}$$

The bore conditions are

$$U = \sqrt{\frac{gh(h_0+h)}{2h_0}}, \tag{3}$$

$$u = \frac{h-h_0}{h}\,U, \tag{4}$$

where U is the bore velocity. Along a positive characteristic it is readily found from (1) and (2) that

$$du + 2dc - \frac{g\,dh_0}{u+c} = 0 \tag{5}$$

on

$$\frac{dx}{dt} = u+c, \tag{6}$$

where $c = \sqrt{(gh)}$. The approximate formula may be derived by applying the differential relation (5) to the flow quantities immediately behind the bore. Since these quantities are given in terms of U by (3) and (4) we have, then, a differential equation for U as a function of h_0. The arbitrary constant in the integration is fixed from the initial value of U.

This simple rule and its motivation are discussed in detail in the previous paper (Whitham 1958). It has not been justified in general, although it is easily shown to be the correct answer in a linearized perturbation theory for the effects of small changes in the depth on a bore initially moving with constant speed in a uniform region. In the linearized theory, the coefficient $(u+c)^{-1}$ in (5) is replaced by its unperturbed value $(u_1+c_1)^{-1}$ say. Then (5) can be integrated to

$$\Delta u + 2\Delta c - \frac{g\Delta h_0}{u_1+c_1} = 0, \tag{7}$$

where Δu, Δc, Δh_0 denote the perturbations $u - u_1$, etc.; the right-hand side of (7) is set equal to zero since all the positive characteristics C_+ come from the initial uniform region in which $\Delta u = \Delta c = \Delta h_0 = 0$ (see figure 1). Since the combination in (7) vanishes on each characteristic, it follows that it vanishes everywhere. In particular, therefore, it applies to the flow quantities just behind the bore. Substituting from the bore conditions we have a relation of the form

$$f(U_1)\,\Delta U = \Delta h_0 \tag{8}$$

for the change in bore speed $\Delta U = U - U_1$ in terms of the change in depth Δh_0. This result is exactly the linearized form of the above rule. The more general form of the rule corresponds to replacing (8) by the differential equation

$$f(U)\frac{dU}{dh_0} = 1. \tag{9}$$

From this point of view, the rule would be expected to hold if the depth varies sufficiently slowly even though the total change in depth is not small. But it turns out to be much better and applies equally well to quite extreme cases. This will be seen in the results presented in this paper.

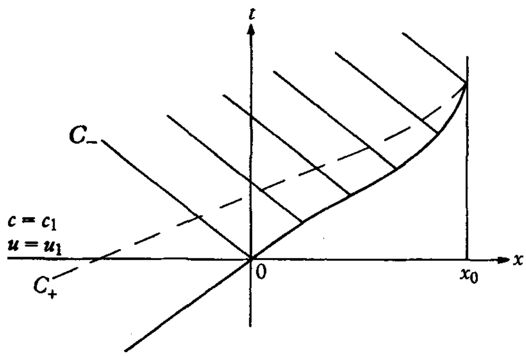

FIGURE 1. The (x, t) diagram for the case $u_1 < c_1$.

In substituting the bore conditions into (5) to get a differential equation corresponding to (9), it is convenient to work with $M = U/\sqrt{(gh)}$. In terms of M, we have

$$\left.\begin{aligned} \frac{U}{\sqrt{(gh_0)}} &= M\sqrt{(2M^2-1)}, \\ \frac{u}{\sqrt{(gh_0)}} &= \frac{2M(M^2-1)}{\sqrt{(2M^2-1)}}, \\ \frac{c}{\sqrt{(gh_0)}} &= \sqrt{(2M^2-1)}, \\ \frac{\eta}{h_0} &= \frac{h-h_0}{h_0} = 2(M^2-1). \end{aligned}\right\} \tag{10}$$

On substitution in (5), the following equation for M as a function of h_0 is obtained:

$$\frac{1}{h_0}\frac{dh_0}{dM} = -4\frac{(M+1)(M-\frac{1}{2})^2(M^3+M^2-M-\frac{1}{2})}{(M-1)(M^2-\frac{1}{2})(M^4+3M^3+M^2-\frac{3}{2}M-1)}. \tag{11}$$

The range of M is $1 < M < \infty$, weak bores corresponding to small values of $(M-1)$ and strong bores to large values of M. In this range, the right-hand side of (11) is always negative so that M increases monotonically as h_0 decreases. However this is not true of the height η. When $M-1$ is small, (11) is approximately

$$\frac{1}{h_0}\frac{dh_0}{dM} = -\frac{4}{5}\frac{1}{M-1},$$

and it gives

$$M-1 \propto h_0^{-\frac{5}{4}}, \quad \eta \propto h_0^{-\frac{1}{4}}. \tag{12}$$

Whereas, when M is large, we have

$$\frac{1}{h_0}\frac{dh_0}{dM} = -\frac{4}{M}, \quad M \propto h_0^{-\frac{1}{4}}, \quad \eta \propto h_0^{\frac{1}{2}}. \tag{13}$$

This is the result noted earlier that the bore height tends to zero when $h_0 \to 0$. For weak bores, η increases as h_0 decreases, but for strong bores η decreases with h_0. Thus in the beach problem, if η is initially less than the critical value $0{\cdot}6262\,h_0$ (see the determination below), the height first increases and then ultimately decreases with a maximum at some intermediate position; at the maximum $\eta/h_0 = 0{\cdot}6262$. If η/h_0 exceeds this value initially, the height decreases all the way to the shoreline. The bore velocity U has a similar change in behaviour, but at a different critical value given by $\eta/h_0 = 2{\cdot}076$, $U/\surd(gh_0) = 2{\cdot}504$. If $U < 2{\cdot}504\,\surd(gh_0)$ initially, U first decreases, but then ultimately increases to a finite value at the shoreline; at the minimum $U/\surd(gh_0) = 2{\cdot}504$. If $U > 2{\cdot}504\,\surd(gh_0)$ initially, U increases all the way to the shore.

The result that $\eta \to 0$ as $h_0 \to 0$ is surprising at first sight and certainly the validity of the rule might be suspected in this extreme limiting case. However, an independent proof can be given as follows:

First, from the bore conditions (3) and (4), the bore height η must tend to zero with h_0 if u is to remain finite. We now show that u must remain finite. For a uniformly sloping beach we may take $h_0 = \alpha(x_0 - x)$, where α is the slope and $x = x_0$ is the shoreline. Then, the characteristic relations (5) and (6) give

$$du + 2\,dc + \alpha g\,dt = 0 \quad \text{i.e.} \quad u + 2c + \alpha g t = \text{constant}, \tag{14}$$

on a positive characteristic. Thus, if $u \to \infty$ at the shoreline, $u + 2c + \alpha g t \to \infty$ at *all* points of the characteristic which reaches the bore just at the shoreline. This is clearly impossible in the solution of our problem. (It should be noted that if $u \to \infty$, $U \to \infty$ so the bore reaches the shoreline in finite time.) Therefore, u remains finite as $h_0 \to 0$; hence, from the bore conditions, $\eta \to 0$. It is interesting to note that this method of fixing the behaviour near the shore, by the absence of a singularity on the characteristic, is completely analogous to the method for determining the exponent in Guderley's similarity solution for the converging cylindrical shock (see the discussion in Whitham (1958)).

The relation (14) can be written in a further convenient form if we assume that the depth is uniform in $x < 0$ and that the bore is initially moving with constant speed in that region. The (x, t) diagram is shown in figure 1 for the case when the initial constant values $u = u_1$, $c = c_1$ behind the bore are such that $u_1 < c_1$. Then the flow is undisturbed to the left of the negative characteristic $x = (u_1 - c_1)\,t$ shown as C_- in the figure. If $u_1 > c_1$ the flow is undisturbed with $u = u_1$, $c = c_1$ in $x < 0$

until a reflected bore appears. In the absence of such secondary bores (which would modify the result somewhat), we have from (14)

$$u+2c+\alpha gt = u^*(\tau)+2c^*(\tau)+\alpha g\tau,$$

where $t=\tau$ is the time when the characteristic crosses $x=0$ and u^*, c^* are the corresponding values of u and c. But, $u+2c$ is constant on the characteristic in $x<0$ since $h_0 =$ constant, therefore $u^*+2c^* = u_1+2c_1$ and we have

$$u+2c+\alpha g(t-\tau) = u_1+2c_1. \tag{15}$$

M	$\dfrac{h_0}{A}=f(M)$	$\dfrac{\eta}{A}$	$\dfrac{U}{\sqrt{(gA)}}$	$\dfrac{c}{\sqrt{(gA)}}$	$\dfrac{u}{\sqrt{(gA)}}$
1·01	5·360	0·2155	2·385	2·361	0·092
1·02	3·006	0·2429	1·839	1·802	0·137
1·03	2·124	0·2586	1·590	1·543	0·173
1·04	1·649	0·2692	1·440	1·385	0·202
1·05	1·350	0·2767	1·339	1·275	0·228
$\sqrt{1\cdot125}$	1·130	0·2825	1·261	1·188	0·252
1·08	$8\cdot696\times10^{-1}$	0·2894	1·163	1·077	0·290
1·1	$6\cdot988\times10^{-1}$	0·2935	1·096	0·996	0·324
1·146	$4\cdot733\times10^{-1}$	0·2964	1·005	0·877	0·387
1·2	$3\cdot342\times10^{-1}$	0·2941	0·951	0·793	0·445
1·3	$2\cdot055\times10^{-1}$	0·2836	0·909	0·699	0·527
1·428	$1\cdot285\times10^{-1}$	0·2668	0·898	0·629	0·606
1·5	$1\cdot028\times10^{-1}$	0·2570	0·900	0·600	0·643
2·0	$3\cdot308\times10^{-2}$	0·1985	0·962	0·481	0·825
$\sqrt{6}$	$1\cdot602\times10^{-2}$	0·1602	1·028	0·420	0·935
3·0	$7\cdot909\times10^{-3}$	0·1265	1·100	0·367	0·035
3·5	$4\cdot637\times10^{-3}$	0·1043	1·155	0·330	1·106
4·0	$2\cdot916\times10^{-3}$	0·0875	1·203	0·301	1·164
5·0	$1\cdot335\times10^{-3}$	0·0641	1·279	0·256	1·253
6·0	$6\cdot992\times10^{-4}$	0·0489	1·337	0·223	1·318
7·0	$4\cdot023\times10^{-4}$	0·0386	1·383	0·198	1·368
8·0	$2\cdot480\times10^{-4}$	0·0312	1·420	0·177	1·408
10·0	$1\cdot094\times10^{-4}$	0·0217	1·476	0·148	1·468
20·0	$8\cdot058\times10^{-6}$	0·0064	1·605	0·080	1·603
∞	0	0	1·763	0	1·763

TABLE 1. Calculations from approximate formula for bore motion: general A

From this form it is immediately obvious that u remains bounded since all terms on the left are positive. However, this form is limited to the particular case of an initially uniform bore and the earlier argument is more general. With this result that u remains bounded, the bore conditions (3) and (4) show that h and η are proportional to $h_0^{\frac{1}{2}}$ as $h_0\to 0$. It should also be noted that the particle velocity u and the bore velocity U approach the same limiting value when $h_0\to 0$.

Turning to the details of the solution given by (11), we see that the maximum of η occurs when

$$\frac{d\eta}{dh_0} = 2(M^2-1)+4h_0 M\frac{dM}{dh_0} = 0. \tag{16}$$

The value of M is found to be 1·146, and from the bore conditions the corresponding η, etc. are

$$\left.\begin{aligned} &\text{Maximum } \eta\text{: } M = 1\cdot146, \quad \eta/h_0 = 0\cdot6262, \quad U/\sqrt{(gh_0)} = 1\cdot461, \\ &\frac{u}{\sqrt{(gh_0)}} = 0\cdot5627, \quad \frac{c}{\sqrt{(gh_0)}} = 1\cdot275. \end{aligned}\right\} \tag{17}$$

(In the earlier paper (Whitham 1958), M was taken erroneously to be approximately 1·2 and this led to a large error in η/h_0). The minimum bore speed occurs at a larger value of M; we have:

$$\left.\begin{aligned} &\text{Minimum } U\text{: } M = 1\cdot428, \quad \eta/h_0 = 2\cdot076, \quad U/\sqrt{(gh_0)} = 2\cdot504, \\ &\frac{u}{\sqrt{(gh_0)}} = 1\cdot690, \quad \frac{c}{\sqrt{(gh_0)}} = 1\cdot754. \end{aligned}\right\} \tag{18}$$

M	$H_0 = \dfrac{h_0}{h_0(0)}$	$N = \dfrac{\eta}{h_0(0)}$	$V = \dfrac{U}{\sqrt{(gh_0(0))}}$	$\dfrac{c}{\sqrt{(gh_0(0))}}$	$v = \dfrac{u}{\sqrt{(gh_0(0))}}$
$\sqrt{1\cdot125}$	1·0000	0·2500	1·186	1·118	0·237
1·08	0·7696	0·2561	1·094	1·013	0·273
1·1	0·6184	0·2597	1·031	0·937	0·305
1·146	0·4189	0·2623	0·946	0·825	0·364
1·2	0·2957	0·2602	0·895	0·746	0·419
1·3	0·1819	0·2510	0·855	0·658	0·496
1·428	0·1137	0·2361	0·844	0·591	0·570
1·5	0·0910	0·2275	0·846	0·564	0·605
2·0	0·0293	0·1757	0·905	0·453	0·776
4·0	0·0026	0·0774	1·131	0·283	1·095
8·0	0·0002	0·0276	1·336	0·167	1·325
∞	0	0	1·659	0	1·659

TABLE 2*a*. Calculations from approximate formula for bore motion: $A = 0\cdot8850$, $N(0) = 0\cdot25$

M	$H_0 = \dfrac{h_0}{h_0(0)}$	$N = \dfrac{\eta}{h_0(0)}$	$V = \dfrac{U}{\sqrt{(gh_0(0))}}$	$\dfrac{c}{\sqrt{(gh_0(0))}}$	$v = \dfrac{u}{\sqrt{(gh_0(0))}}$
$\sqrt{6}$	1·0000	10·000	8·12	3·317	7·39
3·0	0·4936	7·897	8·69	2·897	8·18
3·5	0·2894	6·511	9·13	2·608	8·74
4·0	0·1820	5·459	9·50	2·375	9·19
5·0	0·0833	3·998	10·10	2·020	9·90
6·0	0·0436	3·055	10·56	1·760	10·41
7·0	0·0251	2·410	10·92	1·560	10·81
8·0	0·0155	1·950	11·22	1·402	11·13
10·0	0·0068	1·352	11·66	1·166	11·60
20·0	0·0005	0·401	12·68	0·634	12·66
∞	0	0	13·93	0	13·93

TABLE 2*b*. Calculations from approximate formula for bore motion: $A = 62\cdot41$, $N(0) = 10\cdot0$

Equation (11) integrates to

$$h_0 = A\,\frac{(M^2-\tfrac{1}{2})\exp\{a_1\tan^{-1}(M+a_2)/a_3\}}{(M-1)^{\frac{4}{5}}(M-a_4)^{\alpha}(M^2+a_5M+a_6)^{\beta}(M+a_7)^{\gamma}}, \tag{19}$$

where A is a constant of integration to be determined from the initial strength of the bore, and

$$a_1 = 0\cdot2808, \quad a_2 = 0\cdot6769, \quad a_3 = 0\cdot3179, \quad a_4 = 0\cdot7471, \quad a_5 = 1\cdot354,$$
$$a_6 = 0\cdot5593, \quad a_7 = 2\cdot393, \quad \alpha = 1\cdot180, \quad \beta = 1\cdot173, \quad \gamma = 1\cdot673.$$

The flow quantities at the bore are determined from equations (10); their values are given in table 1. For a given strength of bore initially, M is determined from the bore conditions and then A is determined from (19). The results for $\eta/h_0 = 0\cdot25$ and 10 initially are given in tables 2a and 2b. They are discussed in detail in § 4.

3. Numerical method

It is assumed that the beach has slope α, starts at $x = 0$ and has the shoreline at $x = x_0$; the depth is constant and equal to αx_0 in $x \leqslant 0$. We introduce the dimensionless quantities:

$$\left.\begin{aligned} X &= \frac{x}{x_0}, \quad T = \frac{t}{x_0}\sqrt{(gh_0(0))}, \\ v(X,T) &= \frac{u(x,t)}{\sqrt{(gh_0(0))}}, \quad V = \frac{U}{\sqrt{(gh_0(0))}}, \\ H(X,T) &= \frac{h(x,t)}{h_0(0)}, \quad N(X,T) = \frac{\eta(x,t)}{h_0(0)}, \end{aligned}\right\} \tag{20}$$

where $h_0(0) = \alpha x_0$. In terms of these variables

$$H_0(X) = \begin{cases} 1, & X \leqslant 0, \\ 1 - X, & 0 \leqslant X \leqslant 1. \end{cases} \tag{21}$$

The shallow water theory equations (1), (2) can now be written as

$$H_T + (Hv)_X = 0, \tag{22}$$
$$v_T + \tfrac{1}{2}(v^2)_X + H_X = H_{0X}, \tag{23}$$

and the bore conditions (3), (4) become

$$V = \sqrt{\frac{H(H_0+H)}{2H_0}}, \tag{24}$$

$$v = \left(\frac{H-H_0}{H}\right) V. \tag{25}$$

The numerical procedure is, in outline, to compute the flow quantities behind the bore on a set of net points (X_i, T_k) by means of finite difference approximations of (22) and (23). The bore quantities and its propagation between net points are computed from a special set of difference equations which include the bore conditions (24) and (25), and couple the flow variables at the bore to those behind it by means of the equations of motion. Of course this procedure assumes that the initial state contains a bore and in its present form is not capable of treating problems in which bores develop.

In detail the spatial net is chosen to be uniform, $X_i = i\Delta X$, for convenience, and the time net, $T_{k+1} = T_k + \Delta T_k$, must then satisfy the stability condition stated

below. If ξ_k is the position of the bore at time T_k, we let $X_{s(k)}$ be that net point for which
$$X_{s(k)} \leqslant \xi_k < X_{s(k)+1}.$$
At all net points $X_i < X_{s(k)}$ the flow quantities are computed from the difference forms of (22) and (23) (see figure 2*a*):
$$H(P) = \tfrac{1}{2}[H(R')+H(Q')] - \frac{\Delta T_k}{2\Delta X}[v(R')\,H(R') - v(Q')\,H(Q')], \tag{26}$$
$$v(P) = \tfrac{1}{2}[v(R')+v(Q')] - \frac{\Delta T_k}{2\Delta X}\{\tfrac{1}{2}[v^2(R') - v^2(Q')] + [H(R') - H(Q')]\}$$
$$-\begin{cases} 0, & X_i < 0, \\ \tfrac{1}{2}\Delta T_k, & X_i = 0, \\ \Delta T_k, & X_i > 0. \end{cases} \tag{27}$$

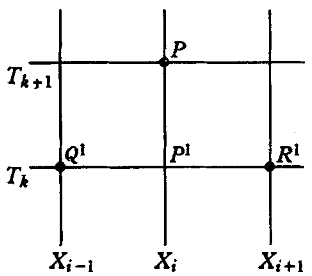

FIGURE 2*a*. Mesh points for differencing the equations.

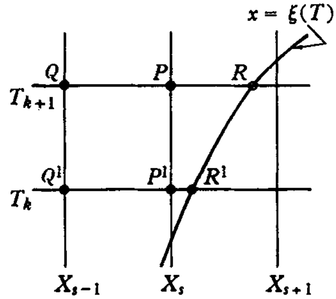

FIGURE 2*b*. Mesh points for fitting in the bore.

These equations are obtained from (22), (23) by replacing the T-derivatives by forward differences and the X-derivatives by centred differences. If in the T-differences the average values $\tfrac{1}{2}[H(R')+H(Q')]$, etc., are replaced by the actual values $H(P')$, etc., the difference equations become unconditionally unstable. However, using these average values it has been shown (Lax 1954) that the equations are stable provided the time mesh satisfies the so-called Courant condition
$$\Delta T_k \leqslant \min_{P'}\left(\frac{\Delta X}{v(P') + \sqrt{\{H(P')\}}}\right).$$

It is clear from (21) that $\partial H_0/\partial X$ is discontinuous at $X = 0$ and as a consequence it follows that $\partial v/\partial X$ and $\partial H/\partial X$ must satisfy the jump conditions
$$\left[\frac{\partial H}{\partial X}\right]_{X=0-}^{X=0+} = \frac{-H}{H-v^2}, \quad \left[\frac{\partial v}{\partial X}\right]_{X=0-}^{X=0+} = \frac{v}{H-v^2}. \tag{28}$$
In these dimensionless variables, $v^2 = H$ when the particle velocity is equal to the propagation speed. In fact this condition never occurs at $x = 0$; if it holds initially, a reflected bore is produced when the main bore meets the change in depth, and this second bore establishes values of v and H, with $v^2 \neq H$, at $x = 0$. This whole question of perturbations on sonic conditions behind a shock has been elucidated by M. P. Friedman (1959). This particular feature of the problem is not under discussion here and we compute only cases with $v^2 - H$ well away from zero.

Again the discontinuity in the bottom and corresponding discontinuities in v_X and H_X are not of primary interest here, since we are mainly concerned with the motion of the bore as it approaches the shoreline. Accordingly no special account of these discontinuities has been taken in (26) and (27). As a consequence we expect and indeed find in the results that the numerical solution suffers from relatively large truncation errors near $X = 0$. These effects could be eliminated either by smoothing the corner in the bottom profile or by modifying the difference equations to include the jump conditions (28). However, the errors introduced in this way at $X = 0$ are not large enough to warrant these detailed corrections for the present purposes.

The net points which enter into the bore fitting procedure are shown in figure 2*b*. All relevant quantities are known at Q, Q', P' and R' and the unknowns are $v(P)$, $H(P)$, $v(R)$, $H(R)$, $V(R)$ and $\xi_{k+1} = \xi(R)$. Thus in order to have a determined system we must add to the equations of motion and bore conditions two additional relations. We use the definition of the bore speed

$$\frac{d\xi(T)}{dT} = V(T), \tag{29}$$

and the bore acceleration, obtained by differentiating (24) and using (22)–(25),

$$\frac{dV(T)}{dT} = \frac{1}{4V}\left\{[VH_X-(vH)_X]\left(1+2\frac{H}{H_0}\right) - VH_{0X}\left(\frac{H}{H_0}\right)^2\right\}. \tag{30}$$

The equations of motion are approximated at P using *backward* T-differences and, for simplicity, X-differences centred at $\frac{1}{2}(Q+R)$ rather than at P. These implicit equations are unconditionally stable and so impose no further restriction on the time mesh. The bore conditions (24) and (25) are to be satisfied at R and the subsidiary equations (29) and (30) are centred on the bore at $\frac{1}{2}(R+R')$. The system of equations obtained in this manner is non-linear and is solved by iterations. In the form and order in which these equations are used they are:

$$\xi(R) = \xi(R') + \tfrac{1}{2}(\Delta T_k)[V(R)+V(R')], \tag{31}$$

$$H(R) = \tfrac{1}{2}\{\sqrt{\{H_0(R)[H_0(R)+8V^2(R)]\}} - H_0(R)\}, \tag{32}$$

$$v(R) = \left(1-\frac{H_0(R)}{H(R)}\right)V(R), \tag{33}$$

$$H(P) = H(P') - \frac{\Delta T_k}{\Delta X + (\xi(R)-X_s)}[v(R)H(R) - v(Q)H(Q)], \tag{34}$$

$$v(P) = v(P') - \frac{\Delta T_k}{\Delta X + (\xi(R)-X_s)}\{\tfrac{1}{2}[v^2(R)-v^2(Q)] + [H(R)-H(Q)]\}$$
$$+\begin{cases} 0, & \xi(R) \leqslant 0, \\ \dfrac{\xi(R)-X_s}{\Delta X + (\xi(R)-X_s)}\Delta T_k, & \xi > 0, \quad X_s < \Delta X, \\ \Delta T_k, & X_s \geqslant \Delta X, \end{cases} \tag{35}$$

$$V(R) = V(R') + \frac{\Delta T_k}{2}[W(R)+W(R')]. \tag{36}$$

Equations (32) and (33) are the bore conditions, equations (34) and (35) are the equations of motion and equations (31) and (36) are the difference forms of (29) and (30). Here $W(R)$ and $W(R')$ are the lengthy but obvious difference forms of the right side of (30). In these evaluations the appropriate X-differences must be extrapolated to R and R'. By this procedure the fitting method yields the correct uniform flow over a uniform bottom. Any other procedure seems to require special considerations in order to obtain this steady solution.

The iterations proceed from an initial estimate of $V(R)$, say $V(R) = V(R')$, in (31). With the $\xi(R)$ thus determined $H_0(R) = H_0(\xi(R))$ is computed from (21) and the quantities in (32)–(36) can then be evaluated in order. This procedure is repeated, using the latest value of $V(R)$ to start each new iteration, until the successive iterates of all quantities are sufficiently close. In actual computations it is found that the number of iterations required to satisfy a prescribed convergence criterion decreases with the mesh widths and initial bore strength, and increases as the bore gets very close to the shoreline.

When the iterations have converged the bore may have crossed one or more net points; i.e. $\xi(R) > X_{s+1}$ in figure 2b. If this is the case, v and H at the intermediate net points are evaluated by linear interpolation between their values at P and R. To aid in smoothing the calculation the time mesh was restricted, in addition to the Courant condition, by

$$\Delta T_k \leqslant \frac{\Delta X}{2V(R')}.$$

In this manner at least two time steps are required for the bore to traverse a mesh width and the above interpolation procedure is used at most every other time step. For all of the calculations reported here it was observed that this latter condition determined ΔT.

4. Discussion of the results

In the calculations a value is chosen for the initial ratio of the bore height to the undisturbed depth, η/h_0, and the other quantities at the bore then follow from the bore conditions. In the first calculations, the bore is started at $x = 0$ (or in some cases one or two mesh lengths to the left) with flow quantities behind the bore constant and equal to the values at the bore. The results are shown here for the cases with initial values $\eta/h_0 = 0{\cdot}25$ and $\eta/h_0 = 10$. The first case is typical of those where η/h_0 is less than the critical value 0·6262 so the height η increases and the bore speed U decreases at first; also the initial flow behind the bore is subsonic, $u_1 < c_1$. The second case is typical of strong bores with η/h_0 above both critical values so that η decreases and U increases all the way to the beach; the initial flow behind the bore is supersonic, $u_1 > c_1$.

The results for the dimensionless bore height $\eta/h_0(0) = N$ and the dimensionless bore speed $U/\sqrt{(gh_0(0))} = V$ are shown in figures 3a and 3b; V_s indicates the value at the shoreline according to the approximate formula. In these calculations, the mesh width was $\Delta X = 0{\cdot}01$. Each calculation of the flow up to the arrival of the bore at the shoreline took about 45 min on a Univac. It should be remembered that in the dimensionless variables the initial undisturbed depth $h_0(0)$ is 1, the shoreline is at $X = 1$ and the slope of the beach is 1. Any other depth $h_0(0)$, distance x_0 or slope α (where $h_0(0) = \alpha x_0$), can be obtained from the scaling in (20), but the initial ratio of bore height to depth is fixed. In the first case, figure 3*a*, a small oscillation appears on the curve of N near $\xi = 0$. This is the effect of the truncation error introduced by the discontinuities in the derivatives at $X = 0$

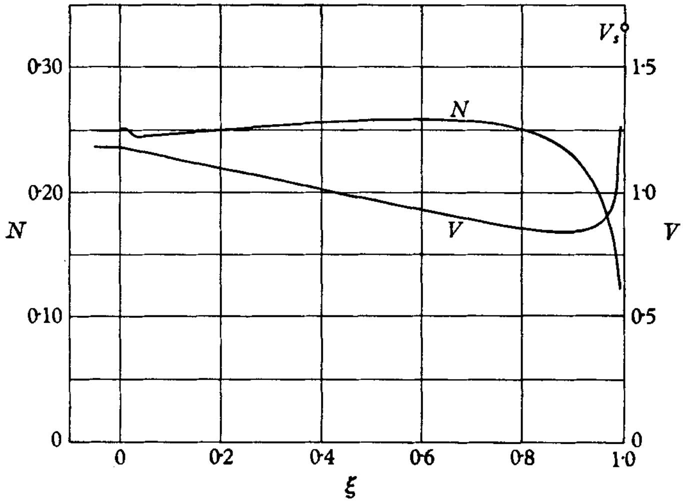

FIGURE 3*a*. Variation of the bore height N and bore velocity V in the case $N(0) = 0{\cdot}25$.

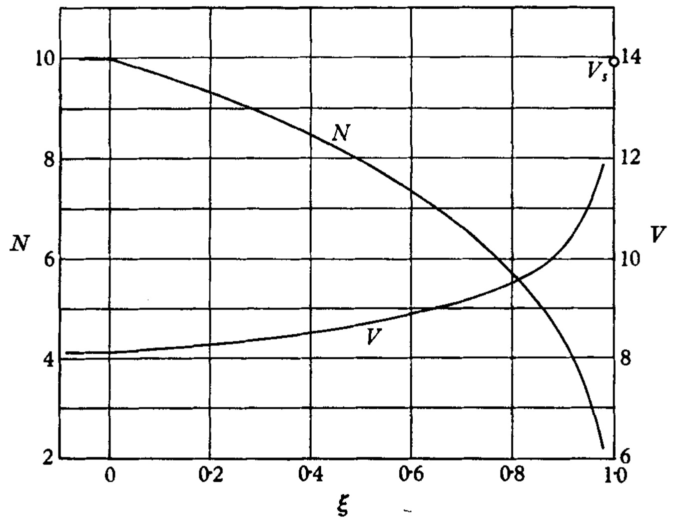

FIGURE 3*b*. Variation of the bore height N and bore velocity V in the case $N(0) = 10{\cdot}0$.

noted in § 3. Corresponding disturbances in the other quantities could not be accurately detected in the numerical results. The observed oscillation in N has a magnitude of about 2 % and is exceptionally large because of the small absolute value of N in this case. As noted in § 3 it did not seem worthwhile to modify the calculation specially to eliminate this particular inaccuracy.

There is no point in plotting the corresponding values given by the approximate formula of § 2 since they lie almost on the curves. Instead the corresponding values are given in tables 2*a* and 2*b*. In the case $N(0) = 0{\cdot}25$, it is interesting to compare the predictions of the approximate formula for the maximum η and minimum U with the numerical calculations. They are shown in table 3. The numerical calculations can be taken with good accuracy to within a mesh width of the shoreline. In fact reasonable results are obtained at $\xi = 0{\cdot}996$. The limiting values of V at $X = 1$ may be obtained by extrapolation; this gives $V = 1{\cdot}568$, for the case $N(0) = 0{\cdot}25$ and $V = 13{\cdot}20$, for the case $N(0) = 10{\cdot}0$. Since V varies very rapidly in the last few mesh widths, we consider also an indirect method of extrapolation derived as follows:

From the characteristic relation (14), we expect $u+2c$ to vary little over the last few mesh lengths because the variation in t is extremely small. Thus, since $c \to 0$, a convenient prediction of u at the shoreline is the value of $u+2c$ (or its

		ξ	N	V	$\dfrac{N}{H_0} = \dfrac{\eta}{h_0}$	$\dfrac{V}{\sqrt{H_0}} = \dfrac{U}{\sqrt{(gh_0)}}$
Maximum η	Approximate formula	0·5811	0·2623	0·9457	0·6262	1·461
	Numerical calculation	0·5938	0·2587	0·9363	0·6367	1·469
Minimum U	Approximate formula	0·8863	0·2361	0·8444	2·076	2·504
	Numerical calculation	0·8835	0·2349	0·8401	2·017	2·462

TABLE 3. Comparison of critical values

extrapolation) over the last few mesh widths. As $h_0 \to 0$, u and U approach the same limiting value. The values obtained by this method for V at the shoreline are $V = 1{\cdot}681$ for the first case, and $V = 13{\cdot}99$ for the second case; the corresponding values from tables 2*a* and 2*b* are 1·659 and 13·93, respectively. In connexion with this extrapolation method it may be noted that the approximate characteristic rule also reduces approximately to $u+2c$ = constant for strong bores.

Turning now to details of the flow behind the bore, the height N and particle velocity v are shown for typical times in figures 4*a* and 4*b*. Again the effect of the truncation error introduced by the discontinuities at $X = 0$ is noticeable in figure 4*a*. In this first case, $u_1 < c_1$ so that a reflected wave moves back into $X \leqslant 0$; in the second case, figure 4*b*, $u_1 > c_1$ so that the flow quantities in $X \leqslant 0$ remain equal to their initial values.

Figures 5*a* and 5*b* show the results of different starting conditions. The initial bore height is kept at $N(0) = 0{\cdot}25$, as for the case shown in figure 3*a*, but behind the bore two different initial distributions of N and v are chosen in place of the constant values. First N and v are taken to fall linearly in $-0{\cdot}1 \leqslant X \leqslant 0$ to $\frac{9}{10}$ of their values at the bore, and then remain constant in $X \leqslant -0{\cdot}1$; secondly, N and v fall in the same distance to $\frac{1}{2}$ their values at the bore. The results for the bore height N and bore speed V are shown as curves 2 and 3 in figures 5*a* and 5*b*, respectively, and compared with the original curves 1 of figure 3*a*. At the points where V is minimum the values of $V/\sqrt{H_0} = U/\sqrt{(gh_0)}$ are 2·462, 2·539, and 2·674, for curves 1, 2 and 3, respectively. These are reasonably close to each other and to the value 2·504 given by the approximate formula in (18). This indicates that the formula (19) still applies near the shore even though the starting conditions are not the ones assumed in that approximate theory. Of course the values of A will be different for the three different cases shown. Curves 1 have A determined from the initial bore height according to § 2. For curves 2 and 3, the approximate formula does not apply in the early stages and some other choice of A must be

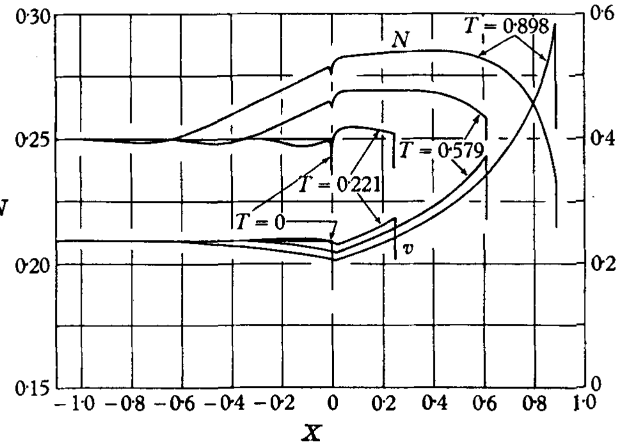

FIGURE 4*a*. The flow behind the bore: height N and particle velocity v in the case $N(0) = 0{\cdot}25$.

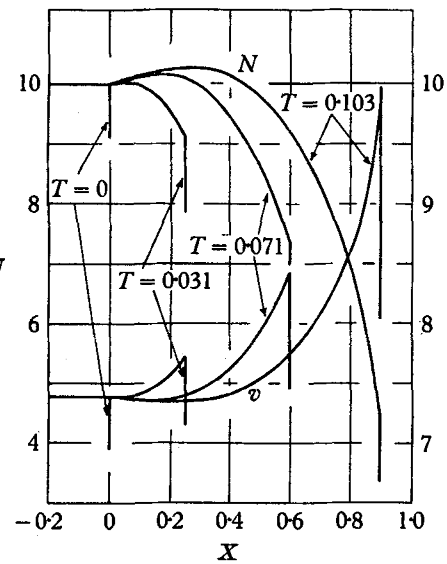

FIGURE 4*b*. The flow behind the bore: height N and particle velocity v v in the case $N(0) = 10{\cdot}0$.

made to see if the formula can describe the curves near the shore. It is convenient to choose A in each case so that the minimum of V agrees. The corresponding curves are drawn dashed in figures 5*a* and 5*b* using the values in table 1 with the appropriate choice of A, which is 0·8004 for curves 2 and 0·5325 for curves 3. These show to what extent the motion of the bore depends on the detailed initial distribution of N and V, before tending to one of the curves (19) as the shoreline is approached. It is interesting to see (figure 5*a*) that in each case the curve goes approximately to the value in the uniform region *behind* the linear profile. This might have been expected of course. It should be remarked that these initial constant values of N and v in $X \leqslant -0{\cdot}1$ do not satisfy the bore conditions exactly; for $N = 0{\cdot}225$, the bore conditions give $v = 0{\cdot}2627$, whereas it was taken to be $0{\cdot}2135$; for $N = 0{\cdot}125$, the bore conditions give $v = 0{\cdot}1367$, whereas it was taken to be $0{\cdot}1186$.

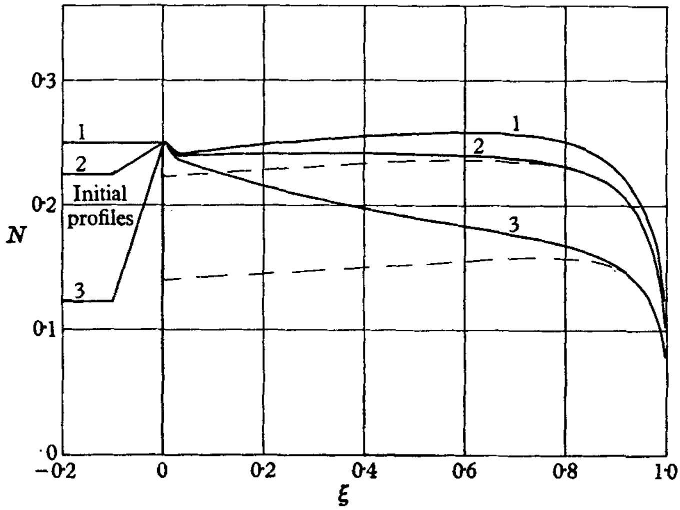

FIGURE 5*a*. Effect of different starting conditions: bore height N.

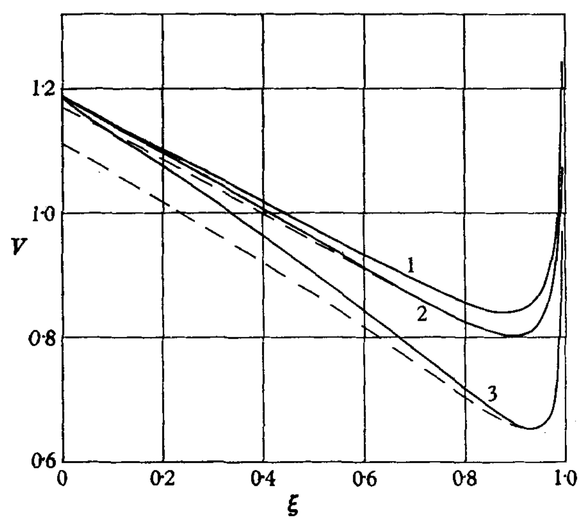

FIGURE 5*b*. Effect of different starting conditions: bore velocity V.

Finally figures 6 and 7 show on a greatly magnified scale, the oscillations due to the truncation errors at $X = 0$ for the case $N(0) = 0{\cdot}25$. These calculations were run until the bore reached $\xi = 0{\cdot}159$ for different mesh sizes and the convergence as $\Delta X \to 0$ is clearly indicated. It was observed (figure 7) that while the mesh size changed by a factor of 8 the value of N at the bore changed by only 4 %. The value $\Delta X = 0{\cdot}01$ was used for the full calculations described above.

All these calculations were performed on the A.E.C. Univac at New York University.

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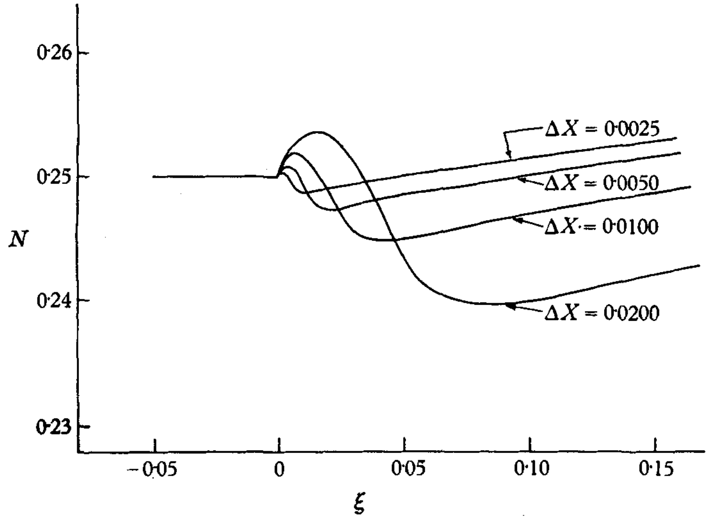

FIGURE 6. Convergence of oscillation in bore height $N(\xi)$.

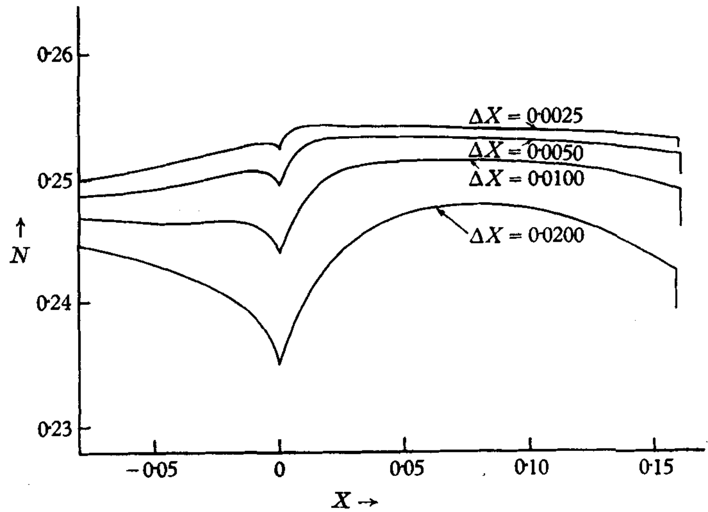

FIGURE 7. Convergence of oscillation in height $N(X, t)$.

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